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FLAT LEFT-INVARIANT CONNECTIONS ADAPTED TO THE AUTOMORPHISM STRUCTURE OF A LIE GROUP

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Suppose that K is a Lie group with Lie algebra $\underline{K}$, and further that $\mathrm{Aut}(\underline{K})$ (respectively $\mathrm{Int}(\underline{K})$) is the group of automorphisms (resp. interior automorphisms) of the algebra $\underline{K}$. The local automorphism (resp. local interior automorphism) structure of K is the principal fiber bundle of frames obtained by the extension to $\mathrm{Aut}(\underline{K})$ (resp. $\mathrm{Int}(\underline{K})$) of a left-invariant parallelism of K. Its fibers are unique, up to a right translation in K's frame bundle $R(K)$. In this article we commence a study of left-invariant locally flat connections adapted to the structures defined above.

INTRODUCTION. PRINCIPAL RESULTS

The problem of finding those Lie groups (necessarily solvable) which admit complete, locally flat (that is, of zero curvature and torsion), left invariant connections is an open problem (cf. J. Milnor [12]). In fact, few groups which possess such connections are known. One of the difficulties encounted while searching for necessary conditions for the existence of such connections is the fact that the relationship between the algebraic structure defined by the connection and that of the Lie algebra is not à priori, sufficiently strong to ensure any consequences for the group structure. Therefore it seems natural to consider, for a first approach to the problem, those connections which are more intrinsic—that is to say—those connections which are adapted to certain left-invariant G-structures over the group, where G is a linear group of automorphisms of the Lie algebra of the group under consideration. Suppose that K is a Lie group with Lie algebra $\underline{K}$, and further that $\mathrm{Aut}(\underline{K})$ (respectively $\mathrm{Int}(\underline{K})$) is the group of automorphisms (resp. interior automorphisms) of the algebra $\underline{K}$. The local automorphism (resp. local interior automorphism) structure of K is the principal fiber bundle of frames of K obtained by the extension

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to Aut($\underline{K}$) (resp. Int($\underline{K}$)) of a left-invariant parallelism of K. Its fibers are unique, up to a right translation in K''s frame bundle $R(K)$.

In this article we commence a study of left-invariant locally flat connections adapted to the structures defined above. The existence of one such connection over K implies that the group K is solvable (§I, 3). This follows from the fact that the Lie product of $\underline{K}$ is given by the commutator of a particular left-symmetric product on $\underline{K}$ which we call a derivation product.

At the base of our results is Theorem 1′ (§II, 2) which is concerned with the structure of the left-symmetric derivation (l.s.d.) algebra. By recalling that flat connections are locally flat connections with trivial holonomy groups, this theorem can be given in terms of Lie groups in the following manner.

Theorem 1. *Suppose that K is a connected and simply connected Lie group with Lie algebra $\underline{K}$. If there exists on K a flat left-invariant connection adapted to the* Aut(K)*-structure, then K can be written as a unique direct product of two normal subgroups K_0 and K_*, which satisfy the following conditions*:

(1) *K_0 is a simply transitive group of affine transformations of the affine space sub-adjacent to the Lie algebra $\underline{K}_0$.*

(2) *The linear components of the action of K_0 on $\underline{K}_0$ are automorphisms of $\underline{K}_0$.*

(3) *K_* is a central subgroup of K.*

(4) *K_* acts by affine transformations on $\underline{K}_*$ leaving one point fixed.*

It is considerably easier to verify the truth of the converse of this theorem. Among the more important consequences of this theorem, one notes the following.

2.3.2. Corollary. *Suppose that K is a connected and simply connected Lie group with Lie algebra $\underline{K}$. If on K there exists a flat left-invariant connection adapted to the* Aut($\underline{K}$)*-structure, then K possesses one such connection which is also complete.*

2.3.4. Corollary. *Suppose that K is a Lie group with Lie algebra $\underline{K}$, that K has a nondiscrete center, and that there exists on K a locally flat left-invariant connection adapted to the* Aut($\underline{K}$)*-structure. If K, considered as a group of affine transformations, acts transitively on $\underline{K}$, then K contains, in its center, nontrivial one-parameter subgroups of translations.*

To better appreciate the importance of the last result, one only has to note that simply transitive groups of affine transformations do not in general contain nontrivial one-parameter subgroups of translations. In [2] L. Auslander has provided an example of this situation, and as well, has conjectured the existence of such subgroups where the group in question is nilpotent. Corollary 2.3.4 gives a proof of the conjecture for the case where the linear components of the elements of the group K are automorphisms of the algebra $\underline{K}$.

We complete this introduction with a few notes concerning the organization and contents of the different sections of this article. The first section is concerned with the relationship between the real or complex left-symmetric algebras of finite dimension and left-invariant affine structures over Lie groups. In the absence of any sufficiently complete reference for this study we have included certain facts which are more or less well known.

The second section constitutes the heart of our work. In the first paragraph we complete the study, undertaken in [4], of locally flat left-invariant connections over K adapted to the Int($\underline{K}$)-structure. In particular, we show that *such connections are complete*. As far as we know this result and Theorem 1.4 (see [4]) give the first systematic examples of simply transitive Lie groups of affine transformations which are not necessarily nilpotent [12]. The second paragraph is concerned with locally flat left-invariant connection adapted to the automorphism structure of a Lie group. In particular this section contains the proof of Theorem 1. Each case is well illustrated with examples.

Finally, in the third section, we concern ourselves with the problem of the existence of those connections studied in this article. We prove that *every Lie group* $\underline{K}$ *with a discrete center and a commutative derived group has a locally flat left-invariant connection adapted to the* Int($\underline{K}$)*-structure*. We exhibit one of these groups which has, up to isomorphism, a single connection of this type. This example permits us to construct solvable Lie groups which do not admit locally flat left-invariant connections adapted to their automorphism structure.

In this article k denotes the field of real or complex numbers. The vector spaces and algebras over k which we consider are of finite dimension. The Lie groups considered are assumed to be connected. For a Lie group K we denote its Lie algebra by $\underline{K}$, and its universal covering group by $\tilde{K}$. By a left-invariant connection on K we shall mean a connection invariant under left-translations.
The author is indebted to Pierre Molino for extremely helpful suggestions.

I. PRELIMINARIES: LEFT-INVARIANT LOCALLY FLAT CONNECTIONS AND LEFT-SYMMETRIC ALGEBRAS

There are very strong ties between the theory of real or complex left-symmetric algebras of finite dimension and the theory of left-invariant affine structures over Lie groups. In this section we briefly recall these ties and interpret various properties of left-symmetric algebras in terms of Lie groups.

In all that follows k is either the field of real numbers or the field of complex numbers.

1. Locally flat manifolds and left-symmetric algebras

Consider a C^∞-differentiable manifold M and the space $\Gamma(TM)$ of sections of its tangent bundle. Let ∇ be a linear connection on M. Then for all elements X and Y of $\Gamma(TM)$ and each couple f and g of differentiable functions on M, we have

$$(1.0) \qquad \nabla_{gX}Y = g\nabla_X Y, \quad \nabla_X(fY) = X(f)Y + f\nabla_X(Y).$$

The two tensor fields C and T defined on M respectively by

$$C(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y],$$
$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

for elements X and Y of $\Gamma(TM)$ are called respectively the curvature and torsion tensors of ∇. The connection ∇ is locally flat if the tensors C and T are identically zero. If, in addition, ∇ has trivial holonomy [18], then as the manifold $M$is assumed to be connected, ∇ is said to be flat. We say that the manifold M is locally flat (resp. flat) if there exists a locally flat (resp. flat) connection on M.

Note that if the tensor field C is zero, then the vector space $\Gamma(TM)$ is an algebra with product $XY = \nabla_X Y$, and has the following property for all elements X, Y and Z of $\Gamma(TM)$:

$$(1.1) \qquad (XY)Z - X(YZ) = (YX)Z - Y(XZ).$$

Any vector space equipped with a bilinear product which satisfies this condition is called a *left-symmetric algebra* [16]. It is easy to verify that if A is a left-symmetric algebra, then A is a Lie algebra under the product $(x, y) \to xy - yx$. This Lie algebra is said to be sub-adjacent to the left-symmetric algebra A.

If ∇ is locally flat, the canonical Lie structure of $\Gamma(TM)$ coincides with the Lie structure sub-adjacent to the product $XY = \nabla_X Y$. We say in this case that the left-symmetric product is compatible with the original Lie structure.

It is evident that M is locally flat if and only if $\Gamma(TM)$ has a left-symmetric product, which is compatible with its canonical Lie structure and satisfies the following conditions

$$(1.0') \qquad (gX)Y = g(XY), \quad X(fY) = X(f)Y + f(XY),$$

for all appropriate elements.

To obtain left-symmetric algebras of finite dimension it suffices to take for M a Lie group K and to suppose that K has a locally flat left-invariant connection. In fact, by identifying the Lie algebra $\underline{K}$ with the left-invariant vector fields on K, we see that $\underline{K}$ is a finite dimensional left-symmetric sub-algebra of the algebra $\Gamma(TK)$. Conversely, if K is such that its Lie algebra

$\underline{K}$ is sub-adjacent to a left-symmetric algebraic structure we find that there exists a locally flat left-invariant connection on K which, because of the properties (1.0′), extends to $\Gamma(TK)$ the product given on $\underline{K}$. If K and $\underline{K}$ are complex, the induced connection on K is adapted to the complex structure. Further, if the left-symmetric product on $\underline{K}$ is associative it is easy to check that the induced connection is also right-invariant. The following result has apparently been known for some time (see, for example, p. 12, p. 186]) but we have not noticed any reference for its proof.

1.1. Proposition. *Let K be a Lie group. Then K has a locally flat bi-invariant connection if and only if its Lie algebra $\underline{K}$ is sub-adjacent to an associative product on K.*

Indeed, when the connection is left-invariant the Lie structure of $\underline{K}$ is compatible with the left-symmetric product. When the connection is bi-invariant, it is also invariant for all interior automorphisms $\tau \to \sigma\tau\sigma^{-1}$ of K. Thus the differential, at the origin, of $L_\sigma \circ R_{\sigma^{-1}}$ is an automorphism of the left-symmetric algebra $\underline{K}$. Consequently we have the following identity for elements a, b and x of $\underline{K}$.

$$\operatorname{ad}_x(ab) = \operatorname{ad}_x(a)b + a\operatorname{ad}_x(b).$$

So we see that the left-symmetric product is associative.

2. Etale affine representations of Lie groups

We consider a vector space V over k and we make the usual identification of the group of affine transformations of V with the semi-direct product $V \times GL(V)$, of the additive group V and the linear group $GL(V)$.

2.1. Definition. Let K be a Lie group, and V a vector space over the same field k. An etale affine representation of K on V is a Lie group homomorphism $\rho\colon G \to V \times GL(V)$ for which there is one point v of V for which the orbit of v for ρ is an open subset of V, and the isotropy group of v for ρ is discrete.

Etale affine representations are a basic tool used in the works of J. L. Koszul and E. B. Vinberg [7], [16] concerning bounded domains and homogeneous convex cones.

Consider an etale affine representation of K over V: $\rho(\sigma) = (Q(\sigma), F_\sigma)$, for all elements σ of K. If $x \to (q(x), f_x)$ is the induced infinitesimal representation, and v is a point of V with open orbit and discrete isotropy, then the vector space homomorphism $x \overset{\psi_v}{\to} q(x) + f_x(v)$ is bijective. For all elements a of $\underline{K}$ we set

$$L_a = \psi_v^{-1} \circ f_a \circ \psi_v. \tag{2.1}$$

It can be immediately verified that the mapping $a \to (a; L_a)$ is an affine representation of the Lie algebra $\underline{K}$, which is isomorphic, in such cases, to the representation $x \to (q(x), f_x)$. Further, as $a \to (a, L_a)$ is a Lie algebra homomorphism of $\underline{K}$ into $V \times gl(V)$, so the Lie structure of $\underline{K}$ is compatible with the left-symmetric product defined by $ab = L_a(b) = R_b(a)$. Consequently a point v with an open orbit and a discrete isotropy for an affine action of G on V determines a locally flat left-invariant connection defined by (2.1). This connection is just the pullback, by the covering mapping, of the connection induced on the orbit of v by the usual connection on V.

Let Ω be the set of points of V with open orbits and discrete isotropies for the action ρ. If v_1 and v_2 are two points of Ω which belong to the same connected component of Ω, then it is easy to see that they define isomorphic left-symmetric products on $\underline{K}$. In particular, if $\Omega = V$, the representation $\rho = (Q, F)$ determines a unique left-symmetric algebraic structure over $\underline{K}$. In the case where k is the field of complex numbers we remark that if an open orbit exists, it is necessarily unique.

If K is simply connected, the existence of a left-symmetric structure over $\underline{K}$, compatible with the Lie structure of $\underline{K}$, is equivalent to the existence of an affine representation of K with a point with an open orbit and a trivial isotropy.

3. Geometric interpretation of some properties of left-invariant algebras

Let $(Q, F): K \to V \times GL(V)$ be a finite dimensional affine representation of a Lie group K. Thus K acts on V by the affine transformations: $\sigma v = Q(\sigma) + F_\sigma(v)$ for elements σ of K and v of V. The following four assertions are seen to be equivalent from the definitions.

(i) There exists a fixed point for the action defined by (Q, F).

(ii) (Q, F) is isomorphic to a linear representation.

(iii) (Q, F) is isomorphic to F.

(iv) Q is a 1-cobord for the linear representation F. That is, there exists an element w of V such that $Q(\sigma) = F_\sigma(w) - w$ for all elements σ of K.

From an infinitesimal point of view the assertions above correspond to the following conditions for the affine representation of the Lie algebra $\underline{K}$: $\underline{K} \to V \times gl(V)$, $(x \to (q(x), f_x))$.

(i) There exists an element a of V such that $q(x) + f_x(a) = 0$ for all elements x of $\underline{K}$.

(ii) (q, f) is isomorphic to a linear representation.

(iii) (q, f) is isomorphic to f.

(iv) q is a 1-cobord for the representation f. That is, there exists an element e of V such that $q(x) = f_x(e)$ for all elements x of K.

We will assume that (Q, F) is etale, and consider a point v of V with an open orbit and a discrete isotropy. From what we have seen in §2, the representation (q, f) is isomorphic to the affine representation of $\underline{K}$: $a \to (a, L_a)$. Therefore we conclude from the above that if (Q, F) leaves a point of V fixed, there exists a point e of V such that $x = L_x(e) = R_e(x)$ for all elements x in $\underline{K}$.

In summary, we can say that a simply connected Lie group $\tilde{K}$ admits an etale affine representation which leaves a point fixed if and only if $\underline{K}$ is sub-adjacent to a left-symmetric product having a right identity.

Now we identify the space V considered as a vector space, with the vector hyperplane $V \times \{0\}$ of $V \times k$, and we identify V, considered as an affine space, with the affine hyperplane $V \times \{1\}$ of $V \times k$. Having made this identification we can regard the affine representation $\rho = (Q, F)$ as a linear representation of G by endomorphism of $V \times k$:

$$\rho(\sigma)(v, \lambda) = \big(F_\sigma(v) + \lambda Q(\sigma), \lambda\big).$$

Evidently $V \times \{1\}$ and $V \times \{0\}$ are invariant under ρ. Let H be a Lie subgroup of K, and suppose that ρ/H is completely reducible. Then there exists a subspace W complementary to $V \times \{0\}$ in $V \times k$, which is invariant under ρ/H, and, à fortiori, W intersects the affine hyperplane $V \times \{1\}$. Therefore the point $p = W \cap (V \times \{1\})$ is a fixed point of H, and so H is contained in the isotropy subgroup of p.

Suppose for an instant that K is semi-simple and that the a affine representation (Q, F) is etale. Then we can conclude from the above [3, Chap. III, p. 286] that the left-symmetric algebras $\underline{K}$, possesses a right identity element e. We thus have that $\mathrm{ad}_e = L_e - R_e$ with $R_e = \mathrm{id}$ where L_e and R_e are left and right multiplications by e. Consequently the traces of these endomorphisms satisfy

$$\mathrm{Tr}(\mathrm{ad}_e) = \mathrm{Tr}(L_e) - \mathrm{Tr}(\mathrm{id}) = \mathrm{Tr}(L_e) - n,$$

where n is the dimension of $\underline{K}$. Then as the derived ideal $\mathfrak{D}(\underline{K})$ of the Lie algebra $\underline{K}$ is equal to $\underline{K}$, and $x \to \mathrm{ad}_x$ and $x \to L_x$ are the Lie algebra homomorphisms, so $\mathrm{Tr}(\mathrm{ad}_e) = \mathrm{Tr}(L_e) = 0 = n$, which is absurd. In conclusion, *a Lie group that is semi-simple will not admit an invariant locally flat connection*. This result has been known by the geometers of Grenoble for some ten years. From an algebraic point of view we can affirm that Lie algebras K sub-adjacent to left-symmetric products are neither semi-simple. More generally J. Helmstetter has shown in [5] that $\mathfrak{D}(\underline{K}) \subsetneq \underline{K}$.

Now consider the case where (Q, F) is transitive. Then the orbit of every point of V is open since it is precisely V. This tells us that for every element v of V, the linear homomorphism $\psi_v: x \to q(x) + f_x(v)$ is surjective. In particular, if v has discrete isotropy, then (q, f) is isomorphic to the representation $x \to (x, L_x)$ with $L_x = \psi_v^{-1} \circ f_x \circ \psi_v$, and consequently, for all elements a of $\underline{K}$, the endomorphism $x \to x + R_a(x)$ of K is an isomorphism. This is the motivation for the following definition.

3.1. Definition. Let A be a left-symmetric algebras over the field k. A is said to be transitive if for all elements a of A the endomorphism of the vector space A given by $x \to x + R_a(x) = x + xa$ is an isomorphism.

If $\tilde{K}$ is the universal covering group of K, then the following assertion is evident.

The action of $\tilde{K}$ on the space V is simply transitive if and only if the Lie structure of $\underline{K}$ is sub-adjacent to a transitive left-symmetric algebras structure.

We remark that if the set Ω of points with open orbits and discrete isotropies for the action (Q, F) coincides with V, then (Q, F) determines a unique left-symmetric product on $\underline{K}$. Further, the assertion mentioned above enables us to conclude that a Lie algebra sub-adjacent to a transitive left-symmetric product is necessarily solvable. In fact, if the action of $\tilde{K}$ is simply transitive, then $\tilde{K}$ and V are diffeomorphic as manifolds; on the other hand, if $\tilde{K}$ is not solvable, then it contains nontrivial compact subgroups, and $\tilde{K}$ is not diffeomorphic to V.

From the point of view of connections. the transitivity of a left-symmetric algebras $\underline{K}$ is equivalent to the fact that the left-invariant associated connection on $\tilde{K}$ is complete.

The problem of finding those Lie algebras which are sub-adjacent to a left-symmetric transitive product is an open problem. In fact, very few such algebras are known [12]; we will give here some new examples.

The following notion was introduced by J. Helmstetter in [5].

3.2. Definition. Let A be a left-symmetric algebras. We say that A is nilpotent if, for all elements a of A, the right multiplication R_a of A is a nilpotent endomorphism.

It is evident that left-symmetric nilpotent algebras are transitive. Conversely, if A is a left-symmetric transitive algebra over an algebraically closed field, then A is nilpotent. In fact, if A is transitive the right multiplications R_a of A have no nonzero eigenvalues.

It may turn out that Definition 3.1 is superfluous. That is to say, all transitive algebras may be nilpotent. However, at the time of writing this article we are unaware of the complete answer to this question. For the algebras we consider below the two notions are equivalent.

We return to the affine representation (Q, F), and assume that it is etale. Suppose that G contains, for the representation, a nontrivial translation of V. Infinitesimally this is just the existence of a nonzero element x of $\underline{K}$ such that $q(x) \neq 0$ and $f_x = 0$. For the affine representation of the Lie algebra $\underline{K}$: $a \to (a, L_a)$ defined for some point of Ω, this tells us that there exists a nonzero element a of $\underline{K}$ such that $L_a = 0$.

Let A be a left-symmetric algebra, and consider the set $N(A) = \{a \in A;\, L_a = 0\}$. Evidently $N(A)$ is a right ideal of A. Thus as $a \to L_a$ is a Lie algebra homomorphism, $N(A)$ is a Lie ideal of A. Therefore $N(A)$ is a bilateral ideal of A. We propose

3.3. Definition. The kernel ideal (or simply, the kernel) of a left-symmetric algebra A is the bilateral ideal of A defined by

$$N(A) = \{a \in A; \forall x \in A,\, ax = 0\}.$$

This being given, we have

For the representation ρ, $\underline{K}$ contains nontrivial one-parameter subgroups of translations if and only if the left-symmetric algebra $\underline{K}$ defined by ρ, has a nonzero kernel ideal.

There exist left-symmetric nilpotent algebras for which the kernel ideal is zero (see [2]). L. Auslander has conjectured, in the language of Lie groups, that a left-symmetric transitive algebra, for which the subadjacent Lie algebra is nilpotent, has a nontrivial kernel ideal. In our opinion this assertion remains to be demonstrated despite the proof given in [14].

II. LEFT-INVARIANT CONNECTIONS ADAPTED TO THE AUTOMORPHISM STRUCTURE. LEFT-SYMMETRIC DERIVATION ALGEBRAS

1. (a) Locally flat left-invariant connections and certain left-symmetric structures

Let K be a Lie group with Lie algebra $\underline{K}$, and let $\mathrm{Aut}(\underline{K})$ and $\mathrm{Int}(\underline{K})$ denote respectively the group of automorphisms and the group of interior automorphisms of the Lie algebra $\underline{K}$. We identify $\underline{K}$ with the left-invariant vector fields on K, and consider a base $B = \{e_1, e_2, \cdots, e_n\}$ of $\underline{K}$. B is a principal fibre bundle of frames of K with trivial structural group. The right action of $\mathrm{Aut}(\underline{K})$ on B defines a bundle of frames of K:

$$P = \{\{e_1, e_2, \cdots, e_n\}\sigma;\, \sigma \in \mathrm{Aut}(\underline{K})\},$$

which is a principal fibre bundle with structure group Aut($\underline{K}$). We say that ρ is the bundle of (local) automorphisms of K defined by the parallelism B of K. The bundle of K given by the restriction of P to the group Int($\underline{K}$) is the bundle of (local) interior automorphisms of K defined by B. The bundles determined by two invariant parallelisms of K are conjugates of each other. We propose

1.1. Definition. Let K be a Lie group with Lie algebra $\underline{K}$. The automorphism structure (resp. the adjoint structure) of K is the principal fibre bundle on K obtained by extension to Aut($\underline{K}$) (resp. to Int($\underline{K}$)) of a left-invariant parallelism of K.

We recall that the structures of Definition 1.1 are unique up to a right-translation in the frame bundle $R(K)$ of K. As the vector fields e_i are invariant under left-translations of K, it is evident that these translations are automorphisms of the structures defined above. Therefore we say that these structures are left-invariant. We remark that the adjoint structure is invariant by interior automorphisms of K and that consequently it is also invariant under right-translations of K.

To our knowledge, P. Molino seems to have been the first author to be interested in the study of these structures. In [13] he demonstrates the transitivity of the adjoint structure. This property is also true for the automorphism structure of K. We also remark that the 0-Cartan-connection on K is adapted to the structures of Definition 1.1; in particular, their first structure functions ("premiers tenseurs de structure") vanish [15].

We consider now locally flat invariant connections on K, adapted to the automorphism structure. We denote by der($\underline{K}$) and ad($\underline{K}$) respectively the Lie algebras of the derivations and the interior derivations of the algebra $\underline{K}$. If ∇ is a left-invariant connection on K, it is evident that ∇ is adapted to the automorphism (resp. interior automorphism) structure of K if and only if the linear mapping $\theta\colon \underline{K} \to \mathrm{gl}(\underline{K})$ defined by $\theta(x) = \nabla_x$ takes its values in the algebra det($\underline{K}$) (resp. ad($\underline{K}$)). We propose

1.2. Definition. A left-symmetric algebra A over k is said to be a derivation (resp. interior derivation) algebra if its left multiplications L_a or its right multiplications R_a are derivations (resp. interior derivations) of the Lie algebra A. In this case we say that the left-symmetric product is a derivation (resp. interior derivation) product.

The Lie group K possesses a left-invariant locally flat connection adapted to the structure of its automorphisms (resp. interior automorphisms) if and only if the Lie algebra $\underline{K}$ is sub-adjacent to a left-symmetric derivation (resp. interior derivation) product.

(b) Locally flat invariant connections adapted to the adjoint structure

1.3. Proposition. *Let A be a Lie algebra over k, and let f be an endomorphism of the vector space A. The product $(a, b) \to ab = [f(a), b]$ on A defines a left-symmetric product compatible with the Lie structure of A if and only if the following two conditions are satisfied for all elements a and b of A:*

$$[a, b] = [f(a), b] + [a, f(b)], \quad [f(a), f(b)] = f[a, b] \pmod{Z(A)},$$

where $Z(A)$ is the center of the Lie algebra A.

The proof of the proposition is immediate.

Study of the endomorphism f produces the following result (see [4]).

1.4. Theorem (*G. Giraud, A. Medina*). *Let K be a Lie group with Lie algebra $\underline{K}$. Then K admits a locally flat left-invariant connection adapted to the adjoint structure if and only if $\underline{K}$ has a decompostion as a direct sum of three vector sub-spaces $\underline{K}_0$, $\underline{K}_1$, $\underline{K}_*$, which satisfy the following conditions*:

$$\begin{aligned} &[\underline{K}_0, \underline{K}_0] = [\underline{K}_1, \underline{K}_1] = \left[\underline{K}_0, \underline{K}_*\right] = \left[\underline{K}_1, \underline{K}_*\right] = 0, \\ &[\underline{K}_0, \underline{K}_1] \subset \underline{K}_0 + Z(\underline{K}), \left[\underline{K}_*, \underline{K}_*\right] \subset Z(\underline{K}). \end{aligned} \tag{1.3}$$

For the details of the demonstration of this theorem see [4] or [11].

1.5. Remark. If there exists an endomorphism f, that decomposes $\underline{K}$ as above, we can modify f so that the supplementary condition, $Z(\underline{K}) \subset \underline{K}_0$, is satisfied [11]. This condition implies that $[\underline{K}_0, \underline{K}_1] \subset \underline{K}_0$.

1.6. Examples. Conditions (1.3) and the above remark imply that $\underline{K}_0$ is an Abelian ideal of $\underline{K}$, which also contains $Z(\underline{K})$ and $\mathfrak{D}(\underline{K}) = [\underline{K}, \underline{K}]$; thus the algebra $\underline{K}$ is 2-solvable. That is to say, $\mathfrak{D}(\underline{K})$ is Abelian.

(a) If $\underline{K}_0$ or $\underline{K}_1$ is zero, then $\underline{K}$ is 2-nilpotent, that is, $\mathfrak{D}(K) \subset Z(\underline{K})$. Conversely, let $\underline{K}$ be 2-nilpotent, and B be a subspace complimentary to $\mathfrak{D}(\underline{K})$ in $\underline{K}$. We obtain a decomposition as in the theorem by setting

$$\underline{K}_0 = Z(\underline{K}), \quad \underline{K}_1 = 0, \quad \underline{K}_* = B,$$

or alternatively

$$\underline{K}_0 = 0, \quad \underline{K}_1 = Z(\underline{K}), \quad \underline{K}_* = B.$$

(b) Every Lie algebra $\underline{K}$, which is a semi-direct product of two Abelian algebras, is sub-adjacent to a left-symmetric interior derivation product. In fact, if $\underline{K} = \underline{K}_0 \times \underline{K}_1$, it suffices to define the product by $ab = [a_0, b]$, where a_0 is the component of a in A_0. The product on $\underline{K}$ defined by $a.b = -ba$ gives $\underline{K}$ an algebraic structure of type P_1 in the sense of [4].

This example enables us to say that every non-semi-simple Lie group of dimension $\leqslant 3$ admits a locally flat left-invariant connection adapted to the adjoint structure.

In §III we will show that every 2-solvable Lie algebra, for which the center is zero, falls in the class of this example. A particular case is given by the Lie k-algebra with base $\{e_1, \cdots, e_n\}$, for which the product is defined by the following rules:

$$[e_1, e_j] = e_j \quad \text{for all } j \geqslant 2,$$
$$[e_i, e_k] = 0 \quad \text{for all } i, k \geqslant 2.$$

(c) There exist 2-solvable algebras which do not satisfy Theorem 1.3 (see [4]).

There also exist 2-solvable Lie groups for which the adjoint structure is not flat in the sense of the theory of G-structures. This is the case for a simply connected group for which the Lie algebra $\underline{K}$, is given by

$$[e_1, e_3] = [e_1, e_4] = [e_2, e_3] = [e_2, e_4] = 0;$$
$$[e_1, e_2] = e_5, [e_1, e_5] = e_3, [e_2, e_5] = e_4;$$
$$[e_i, e_j] = 0 \text{ for } i, j \geqslant 3.$$

In fact, a direct calculation shows that the first prolongation $(\mathrm{ad}(\underline{K}))^{(1)}$ (see [15] for the definition—"l'espace de prolongement") vanishes. This example was provided by G. Giraud.

We now consider the problem of finding when the connections of Theorem 1.4 are complete.

1.7. Theorem. *Every locally flat left-invariant connection adapted to the adjoint structure of a Lie group K is complete.*

Under the hypothesis of the theorem the Lie algebra $\underline{K}$ of the group K is sub-adjacent to a left-symmetric product of type $ab = [f(a), b]$. From §I.3, the connection defined by this product is complete if and only if the left-symmetric algebra $\underline{K}$ is transitive. We prove this theorem by showing that the left-symmetric algebra $\underline{K}$ is nilpotent (Lemma 1.9 below).

We begin with the following proposition.

1.8. Proposition. *Let A be a left-symmetric algebra over a field k. Then the following four assertions are equivalent*:

(a) *for all elements a of A, the left-multiplication L_a is a derivation of the Lie algebra A,*

(b) *for all elements a of A, the right-multiplication R_a is a derivation of the Lie algebra A,*

(c) *for all elements a of A, the bilinear mapping on A $(b, c) \to (ba)c$ is symmetric,*

(d) *for all elements a and b of A, $R_b R_a = L_{ba}$.*

The demonstration of the proposition is trivial: in fact, in every left-symmetric algebra we have the identities $L_a - R_a = \mathrm{ad}_a$;

$$a[b, c] - [ab, c] - [b, ac] = (ca)b - (ba)c,$$

for any elements a, b and c of A.

1.9. Lemma. *Let A be a left-symmetric algebra over k. If the left-multiplications L_a are interior derivations of the Lie algebra A, then the left-symmetric algebra is nilpotent.*

Demonstration. Let f be an endomorphism of the vector space A such that $L_a = \mathrm{ad}_{f(a)}$ for all elements a of A. Theorem 1.4 tells us that the Lie algebra is solvable; in fact, A is 2-solvable. Let N be the maximal nilpotent ideal of the Lie algebra A; N is determined by the fact that $N = \{a \in A; \mathrm{ad}_a \text{ nilpotent}\}$. Further, N contains the derived Lie ideal $\mathfrak{D}(A) = [A, A]$.

Using Proposition 1.8 we see that for all elements a of A, $(R_a)^2 = L_{a^2} = \mathrm{ad}_{f(a^2)}$. So by Proposition 1.3, $f(a^2) = f[f(a), a] \in Z(A) + \mathfrak{D}(A)$, and consequently $\mathrm{ad}_{f(a^2)}$ is nilpotent. As the endomorphism $(R_a)^2$ is nilpotent, R_a is also so.

This finishes the demonstration of the theorem.

Note that if A is an algebra as those of the lemma, and B is a Lie ideal of A (resp. a Lie sub-algebra invariant under f), then the fact that $L_a = \mathrm{ad}_{f(a)}$ for all elements a of A implies that B is a bilateral ideal (resp. a sub-algebra) of the left-symmetric algebra. In particular, this proves

1.10. Proposition. *A locally flat left-invariant connection adapted to the adjoint structure of the Lie group K induces a connection of the same nature on every normal subgroup of K.*

To finish this paragraph we include with the following result:

1.11. Proposition. *Let K be a Lie group. Then K possesses a locally flat bi-invariant connection adapted to the adjoint structure if and only if K is 2-nilpotent, that is, if and only if the derived group of K is central.*

Demonstration. Let $\underline{K}$ be the Lie algebra of K. If K is 2-nilpotent, then the derived Lie ideal $\mathfrak{D}(\underline{K}) = [\underline{K}, \underline{K}]$ is contained in $Z(\underline{K})$, and thus the mapping $\underline{K} \to \underline{K} \times \mathrm{ad}(\underline{K})$, $x \to (x, \frac{1}{2}\mathrm{ad}_x)$, is an affine representation of the algebra $\underline{K}$. The induced connection on K by this representation is the 0-Cartan-connection. Further, the fact that the ideal $\mathfrak{D}(\underline{K})$ is central implies, via the Jacobi identity, that $L_{ab} = L_a L_b$ for all elements a and b of $\underline{K}$, where $L_a = \frac{1}{2}\mathrm{ad}_a$. Thus the left-symmetric algebra $\underline{K}$ is associative, and so the connection is bi-invariant (Proposition 1.1, §1).

Conversely, suppose that K has a connection as that of the theorem. Then there exists a decomposition $\underline{K} = \overline{K}_0 + \overline{K}_1 + \overline{K}_*$ of the vector space $\underline{K}$ which satisfies conditions (1.3) with $Z(\underline{K}) \subset \overline{K_0}$. The linear endomorphism f of $\underline{K}$ given by $f(a) = a_1 + \frac{1}{2}a_*$ defines a locally flat left-invariant connection adapted to the adjoint structure of K. We set $ab = [f(a), b]$ for all elements a and b of $\underline{K}$; evidently $(ab)c = 0$ for all elements a, b and c of $\underline{K}$. Thus this product is associative if and only if for all elements a, b and c of $\underline{K}$ we have $a(bc) = 0$. Then $a(bc) = [a_1, [b_1, c_0]] = 0$, and $[b, c] = [b_0, c_1] + [b_1, c_0] + [b_*, c_*]$ imply that $[a, [b, c]] = 0$ for all elements a, b, and c of $\underline{K}$.

2. Flat left-invariant connections adapted to the automorphism structure

Let K be a Lie group with Lie algebra $\underline{K}$. From the preceding paragraph, we know that the existence of a (locally) flat left-invariant connection on K adapted to the automorphism structure is equivalent to the existence of a left-invariant derivation product compatible with the Lie product of $\underline{K}$.

2.1. Examples of left-invariant derivation algebras.

(a) Let A be a left-symmetric commutative algebra. The Lie algebra subadjacent to A is Abelian, and so for all elements a of A, the left-multiplication L_a or the right-multiplication R_a is a derivation of the Lie algebra A. Thus A is a left-symmetric derivation algebra. We remark in passing that the algebra A is also associative. In fact, $[L_a, L_b] = L_{[a,b]} = 0$ for all elements a and b of A, and consequently

$$(ab)c = b(ac), \quad a(bc) = b(ac)$$

for all appropriate elements. Thus we have

$$(ab)c - a(bc) = c(ab) - a(cb) = c(ab) - c(ab) = 0.$$

This example shows that there exist left-symmetric algebras which are not transitive. We also note that the derivations L_a are interior if and only if the algebra A has a trivial product.

(b) Let A be a vector space over k with the basis $\{e_i\}$, $1 \leqslant i \leqslant 5$. We consider the bilinear product defined on A by

$$L_{e_1} = L_{e_2} = 0, \quad L_{e_3}(e_1) = L_{e_3}(e_2) = -e_1, \quad L_{e_3}(e_4) = L_{e_3}(e_5) = e_1,$$
$$L_{e_4}(e_1) = L_{e_5}(e_1) = L_{e_4}(e_2) = L_{e_5}(e_2) = -e_1,$$
$$L_{e_4}(e_3) = L_{e_5}(e_3) = e_1 - e_2,$$
$$L_{e_4}(e_4) = \tfrac{1}{2}(e_1 + e_2) = L_{e_5}(e_4) = L_{e_4}(e_5) = L_{e_5}(e_5),$$

with the other products defined to be zero. It can be directly verified that this product is left-symmetric. The product of the Lie algebra, sub-adjacent to the above algebra satisfies

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1 \text{-} [e_1, e_4];$$
$$[e_1, e_5] = [e_2, e_3] = [e_2, e_4] = [e_2, e_5] = e_1;$$
$$[e_3, e_4] = [e_3, e_5] = e_2; \quad [e_4, e_5] = 0.$$

We remark that L_{e_3} and $L_{e_4} = L_{e_5}$ are exterior derivations of the Lie algebra A; consequently A is a left-symmetric derivation algebra. Note also that A is a left-symmetric nilpotent algebra, and that the Lie ideal $\mathfrak{D}(A)$ admits a complimentary sub-space S in A, which satisfies the condition $[S, S] \subset Z(A)$.

(c) Let A be a 2-solvable Lie algebra—that is, a Lie algebra such that the derived ideal $\mathfrak{D}(A)$ is Abelian. Suppose further that A decomposes as a direct sum of sub-spaces $A = \mathfrak{D}(A) \oplus S$ with $[S, S] \subset Z(A)$. For elements a of A, we denote by $a_{\mathfrak{D}}$ and a_S the respective components of a in $\mathfrak{D}(A)$ and S. Let f be the endomorphism of the vector space A defined by $f(a) = a_{\mathfrak{D}} + \frac{1}{2} a_S$ for all elements a of A. Then we set

$$L_a = \mathrm{ad}_{a_S} \circ f.$$

A direct calculation shows that the product $ab = L_a(b)$ on A defines a left-symmetric derivation algebra compatible with the Lie structure of A.

A particular case of this example is given by the Lie algebra a over k with basis $\{e_i\}$, $1 \leqslant i \leqslant 5$, defined by the following products:

$$[e_i, e_j] = 0, \qquad 1 \leqslant i, j \leqslant 3; \quad [e_1, e_4] = e_1, \quad [e_2, e_4] = e_2,$$
$$[e_3, e_4] = e_3, \quad [e_1, e_5] = -e_1, \quad [e_2, e_5] = [e_3, e_5] = [e_4, e_5] = 0.$$

(d) Let A be the Lie algebra over k with basis $\{e_i\}$, $1 \leqslant i \leqslant 5$, defined by

$$[e_i, e_j] = \begin{cases} e_{i+j} & \text{if } i < j \text{ and } i + j \leqslant 5, \\ 0 & \text{if } i + j > 5. \end{cases}$$

On this basis the matrix of a derivation of the algebra A is given by

$$D = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 \\ \beta & 2\alpha & 0 & 0 & 0 \\ \delta & \gamma & 3\alpha & 0 & 0 \\ \varepsilon & \omega & \gamma & 4\alpha & 0 \\ \nu & \eta & \omega - \delta & \beta + \gamma & 5\alpha \end{pmatrix}.$$

Knowing these derivations, we can find all the left-symmetric derivation products compatible with the Lie structure of A. Essentially there are two families of such products, of which one is

$$L_{e_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \delta & 1 & 0 & 0 & 0 \\ \varepsilon & \omega & 1 & 0 & 0 \\ \nu & \eta & \omega - \delta & 1 & 0 \end{pmatrix}, \qquad L_{e_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \omega & 1 & 0 & 0 & 0 \\ \eta & \mu & 1 & 0 & 0 \end{pmatrix},$$

$$L_{e_3}(e_1) = (\omega - \delta)e_5,$$

with all the other products being zero, and the coefficients δ, ε, ν, ω, η and μ belonging to k.

We consider the derivation D of the algebra A defined by $D(e_i) = ie_i$, $1 \leqslant i \leqslant 5$. The product $ab = L'_a b$ on A, where $L'_a = D^{-1} \circ \operatorname{ad}_a \circ D$, makes A a left-symmetric algebra. This product is compatible with the Lie structure of A but the multiplications L_a are not all derivations of the Lie product. For example, we have

$$L'_{e_1}[e_1, e_2] = L'_{e_1}(e_3) = D^{-1}[e_1, 3e_3] = \tfrac{3}{4}e_4,$$

whereas

$$\left[L'_{e_1}e_1, e_2\right] + \left[e_1, L'_{e_1}e_2\right] = [0, e_2] + \left[e_1, \tfrac{2}{3}e_4\right] = \tfrac{2}{3}e_5.$$

(e) Let A be the Lie algebra of dimension 5 over k defined by the following identities:

$$\begin{aligned} &[e_1, e_3] = e_5, \quad [e_2, e_3] = 0, \quad [e_3, e_4] = 0, \\ &[e_1, e_3] = e_3, \quad [e_2, e_4] = e_4, \quad [e_5, A] = 0, \\ &[e_1, e_4] = 0. \end{aligned}$$

A left-symmetric derivation product on A compatible with the Lie structure is obtained by taking for the left-multiplications the following endomorphisms:

$$L_{e_1} = \operatorname{ad}_{e_1}, \quad L_{e_2} = \operatorname{ad}^2_{e_2}, \quad L_{e_3} = L_{e_4} = L_{e_5} = 0.$$

Consider the Lie algebra (extension of A) $A' = A \times ke_6$ of dimension obtained from A by imposing

$$[e_1, e_5] = e_6, \quad [e_i, e_6] = 0 \quad \text{for } 1 \leqslant i \leqslant 6.$$

The endomorphisms of A',

$$L'_{e_1} = \operatorname{ad}'_{e_1}, \quad L'_{e_2} = \left(\operatorname{ad}'_{e_2}\right)^2, \quad L'_{e_3} = L'_{e_4} = L'_{e_5} = L'_{e_6} = 0,$$

define on A' a left-symmetric derivation product compatible with the Lie structure.

By a series of such extensions we can obtain a Lie algebra, of any order of nilpotency, sub-adjacent to a left-symmetric derivation product.

Observations. ① We are unaware of any example of a Lie algebra with an order of solvability greater than 2, which has, as its product, the commutator of a left-symmetric derivation product.

② For other examples see §3 below and [11].

2.2. Elementary properties of left-symmetric derivation algebras.

2.2.1. A l.s.d. algebra with a left (or right) identity is commutative and associative.

In fact, if there exists an element e of A such that $L_e = I$, then the identity mapping is a derivation of the Lie algebra A, and so A is commutative.

2.2.2. Every characteristic Lie ideal of a l.s.d. algebra is a bilateral ideal of the algebra.

An ideal of a Lie algebra A is characteristic if it is invariant under every derivation of A. As the algebra A is l.s.d., all the left-multiplications L_a and all the right-multiplications R_a of A leave such an ideal invariant.

2.2.3. Every Lie algebra sub-adjacent to a l.s.d. algebra is solvable.

This results from the following facts:

(i) The radical of a Lie algebra, A, is a characteristic ideal and hence a bilateral ideal of A.

(ii) There exist no semi-simple Lie algebras sub-adjacent to a left-symmetric product (see §I, 3).

2.2.4. A left-symmetric algebra S is a l.s.d. algebra if and only if for all elements a and b of A we have $L_{ab} = R_a \circ R_b$ (see Proposition 1.8, §I).

2.2.5. For a l.s.d. algebra the following identities are true:

$$L_{[a,b]} = [L_a, L_b] = [R_a, R_b].$$

The first identity defines the left-symmetric algebra, and the second recalls 2.2.4.

2.2.6. For a l.s.d. algebra, we have, for all elements a, b, c and d of A,

$$(ab)(cd) = (ac)(bd).$$

In fact, using 2.2.4 we obtain $(ab)(cd) = \{(cd)b\}a = \{(bd)c\}a = (ac)(bd)$.

2.2.7. The center of a l.s.d. algebra coincides with the center of the sub-adjacent Lie algebra and consequently is a commutative and associative bilateral ideal.

Recall that the center of an algebra A consists of those elements z of A, which commute with A, and for which the following associaters are zero:

$$(z, a, b), (a, z, b), (a, b, z),$$

for all elements a and b of A. Since for a left-symmetric algebra we have

$$(a, b, z) = [z, ab] - [z, a]b - a[z, b],$$

so in order to verify 2.2.7 it suffices to show that $(z, a, b) = 0$ for all elements z of the center $Z(A)$ of the Lie algebra A and a and b of A. After 2.2.4 we have

$$(z, a, b) = (za)b - z(ab) = (az)b - (ab)z$$
$$= (bz)a - (zb)a = (bz)a - (bz)a.$$

2.2.8. If $Z(A)$ is the center of a l.s.d. algebra, then $Z(A)\mathfrak{D}(A) = 0$. In fact, for all elements z of $Z(A)$ and a and b of A we have from 2.2.7 and 2.2.4, that

$$z(ab) = (za)b = (ba)z = z(ba).$$

2.2.9. For a l.s.d. algebra the following identity is true:

$$(bc)a^2 = (cb)a^2.$$

From 2.2.4 it follows that $R_a^2 = L_{a^2}$ is a derivation of the Lie algebra A; however R_a^2 is a derivation if and only if $[ba, ca] = 0$ for all elements a, b and c of A, so $(ba)(ca) = (ca)(ba)$. Then 2.2.9. results immediately from 2.2.6.

2.3. A structure theorem and its corollaries. The object of this section is to present our principal result and to give the demonstrations of some of its consequences.

Theorem 1′. *A left-symmetric derivation algebra A over k has a unique decomposition as a direct sum of bilateral ideals, A_0 and A_*, which satisfy the following conditions*:

(1) *A_0 is a nilpotent algebra containing the derived Lie ideal*

$$\mathfrak{D}(A) = [A, A],$$

(2) *A_* is an algebra with an identity and is contained in the center*

$$Z(A) \text{ of } A.$$

Moreover, the kernel ideal $N(A)$ of A vanishes if and only if A_0 does, and $N(A)$ is contained in A_0 when A_0 is not just zero.

Observation. Note that the theorem shows that every non-nilpotent l.s.d. algebra contains an idempotent in the sense of A. Albert: in fact the identity element e of A_* is one such idempotent [1, Lemma 9, p. 25]. Further, as every element a of A can be written as

$$a = (a - ae) + ae$$

with $(a - ae) \in A_0$ and $ae \in A_*$, and the subspaces A_0 and A_* are bilateral ideals of A, the decomposition of A, $A = A_0 \oplus A_*$ can be called a Pierce decomposition of A, [1, p. 24].

2.3.1. Theorem 1. *Let K be a simply connected Lie group with Lie algebra $\underline{K}$. Suppose there exists on K a locally flat left-invariant connection adapted to the* Aut($\underline{K}$)-*structure. Then K has a unique decomposition as a direct product of two normal subgroups K_0 and K_* of K, which satisfy the following conditions*:

(1) *K_0 is a simply transitive group of affine transformations of the affine space sub-adjacent to its Lie algebra $\underline{K}_0$.*

(2) *The linear components of the action of K_0 on $\underline{K}_0$ are automorphisms of the Lie algebra, $\underline{K}_0$.*

(3) *K_* is a central subgroup of K.*

(4) *If $\underline{K}_*$ is the Lie algebra of K_*, then the group K_* acts by affine transformations of $\underline{K}_*$ which leave one point fixed.*

Demonstration of Theorem 1. Under the hypothesis, the Lie algebra $\underline{K}$ is sub-adjacent to a l.s.d. product, and there exists an etale affine representation $\rho: K \to \underline{K} \times \mathrm{Aut}(\underline{K})$ of K. Using Theorem 1′ we see that the algebra $\underline{K}$ decompose as a direct sum of two bilateral ideals $\underline{K}_0$ and $\underline{K}_*$ of $\underline{K}$. Let K_0 and K_* be two simply connected subgroups of K having respectively the Lie algebras $\underline{K}_0$ and $\underline{K}_*$. Evidently K is the direct product of K_0 and K_*. Also, as $\underline{K}_0$ and $\underline{K}_*$ are Lie ideals of $\underline{K}$, and $\underline{K}_*$ is central, so the groups K_0 and K_* are normal subgroups of K, and K_* is contained in the center of K.

Further, as the left-symmetric derivation sub-algebra $\underline{K}_0$ (resp. $\underline{K}_0$) is nilpotent (resp. has an identity), so the group K_0 (resp. K_*) is a simply transitive group (resp. a group leaving one point fixed) of affine transformations of $\underline{K}_0$ (resp. $\underline{K}_*$) (§I, 3).

2.3.2. Corollary. *Let K be a Lie group with the properties in Theorem* 1. *Then K admits a flat left-invariant complete connection adapted to the* $\mathrm{Aut}(\underline{K})$*-structure.*

Demonstration. Under the hypothesis the product of the Lie algebra $\underline{K}$ is the commutator of a l.s.d. product; nevertheless, this product is not, à priori, transitive. With the aid of Theorem 1′ we can change the product in order to obtain one which does enjoy this property. Consider the decomposition $\underline{K} = \underline{K}_0 \oplus \underline{K}_*$ given by Theorem 1′. For elements a and b of $\underline{K}$ we denote their product by ab, and for an element x, of $\underline{K}$ we denote its components in $\underline{K}_0$ and $\underline{K}_*$ respectively by x_0 and x_*. For all elements a and b of $\underline{K}$ we define

$$a_0 \square b_0 = a_0 b_0, \quad a_* \square b_* = 0$$
$$a_0 \square b_* = 0, \quad a_* \square b_0 = 0.$$

It is immediately verified that these relations define a nilpotent l.s.d. product compatible with the Lie structure of $\underline{K}$.

2.3.3. Corollary. *Let K be a Lie group with the properties in Theorem* 1. *Then K, considered as a group of affine transformations of $\underline{K}$, contains nontrivial one-parameter subgroups of translations if and only if there is no point of $\underline{K}$ invariant under K.*

Demonstration. We commence by recalling that the existence of a fixed point, for the representation of K induced by the infinitesimal affine representation $a \to (a, L_a)$ of $\underline{K}$, is equivalent to the existence of an element e, of $\underline{K}$ such that $R_e = I_{\underline{K}}$. Consider the decomposition $\underline{K} = \underline{K}_0 \oplus \underline{K}_*$ given by Theorem 1′. If $R_e = I_{\underline{K}}$, then $\underline{K}_0 = 0$ since $\underline{K}_0$ is nilpotent, and $N(\underline{K}) = 0$ by Theorem 1′,

Conversely, if $N(\underline{K}) = 0$, then Theorem 1′ tells us that $K = K_*$.

2.3.4. Corollary. *Let K be a Lie group with the properties in Theorem* 1, *and suppose that K acts transitively on $\underline{K}$, that is, suppose the connection is complete. Then we have*

(1) *If K has a non-discrete center, then K contains, in its center, non-trivial one-parameter subgroups of translations.*

(2) If K is not commutative, and $Z(\mathfrak{D}(\underline{K}))$ is the center of the Lie ideal $\mathfrak{D}(A)$, then $N(\underline{K}) \cap Z(\mathfrak{D}(\underline{K})) \neq 0$.

Demonstration. From the hypothesis and Theorem 1′ it follows that the Lie algebra $\underline{K}$ is sub-adjacent to a transitive l.s.d. product. Thus under these conditions the two assertions are direct consequences of Engel's theorem.

To prove the first assertion it suffices to apply Engel's theorem to the linear representation $a \to R_a|_{Z(\underline{K})}$ of Lie algebra $\underline{K}$, where $Z(\underline{K})$ is the center of $\underline{K}$. To prove the second assertion we consider the ideal $I = Z(\mathfrak{D}(\underline{K}))$ of the Lie algebra $\mathfrak{D}(\underline{K})$. As $\underline{K}$ is a solvable Lie algebra, $\mathfrak{D}(\underline{K})$ is nilpotent, and hence I is not just the zero element. The Jacobi identity shows that I is a Lie ideal of $\underline{K}$, and it is directly verified that I is a characteristic ideal. Thus I is a bilateral ideal of $\underline{K}$. Further, for all elements z of I, and a and b of $\underline{K}$, we have, from the definition of I, that $R_{[a,b]}z = L_{[a,b]}z$, and by 2.2.5 we also have $[L_a, L_b] = [R_a, R_b]$. These identities imply that the mapping $b \to R_{b|I}$ is a linear representation of the Lie algebra $\underline{K}$.

2.4. Preliminary for the demonstration of Theorem 1′. In order to prove the theorem we will require several lemmas. In particular we use the result of Fitting [6] concerning the decomposition of a vector space relative to an endomorphism. At the base of our demonstration we have the following lemma.

2.4.1. Lemma. *Let A be l.s.d. algebra of finite dimension n over the field k. For all elements a of A we set $A_0(a) = \ker(R_a)^n$ and $A_*(a) = \operatorname{Im}(R_a)^n$. Then $A_0(a)$ and $A_*(a)$ are Lie ideals of A, which satisfy the following conditions*:

$$A = A_0(a) \oplus A_*(a); \quad \mathfrak{D}(A) = [A, A] \subset A_0(A); \quad A_*(a) \subset Z(A).$$

Demonstration. From Fitting's lemma we know that $A = A_0(a) \oplus A_*(a)$, and that the sub-spaces $A_0(a)$ and $A_*(a)$ are invariant under the endomorphism, R_a.

(i) Consider $A_0(a)$. For an arbitrary element d of $\mathfrak{D}(A)$ we know from 2.2.9 that $a^2d - da^2 = [a^2, d] = a^2d$ for all elements a of A. Then, since L_a is a derivation of the solvable Lie algebra A, the element $a^2 = L_a(a)$ belongs to the nilpotent maximal Lie ideal of A, [6]. Thus the identities $(R_a)^2(d) = L_{a^2}(d) = \mathrm{ad}_{a^2}(d)$ (see 2.2.4), imply that the restriction of $(R_a)^2$ (and hence that of R_a) to $\mathfrak{D}(A)$ is a nilpotent endomorphism. Fitting's lemma then implies that $\mathfrak{D}(A) \subset A_0$.

(ii) Consider $A_*(a)$. Here we may assume that k is algebraically closed. For all elements λ of $k^* = k - \{0\}$, define $A_\lambda = \ker(R_a - \lambda I)^n$. Then $A_*(a) = \oplus_\lambda A_\lambda$. As R_a is a derivation of the Lie algebra A, so $[A_\lambda, A_\mu] \subset A_{\lambda+\mu}$. As $(R_a)^2 = L_{a^2}$ is also a derivation, so if $[A_\lambda, A_\mu] \neq 0$, then necessarily $\lambda^2 + \mu^2 = (\lambda + \mu)^2$ and thus $\lambda\mu = 0$. Hence this gives $[A_*(a), A_*(a)] = 0$. Further, the condition $\mathfrak{D}(A) \subset A_0(a)$ implies that $[A_0(a), A_*(a)] \subset A_0(a)$ while $[A_\lambda, A_\mu] \subset A_{\lambda+\mu}$ implies that $[A_0(a), A_*(a)] \subset A_*(a)$. Thus we can conclude that $A_*(a) \subset Z(A)$. q.e.d.

In order to demonstrate the existence of the decomposition of A, it is natural, having seen the above lemma, to determine whether the Lie algebra of endomorphisms of the vector space A generated by the right multiplications R_a is nilpotent. In fact, we find

2.4.2. Lemma. *Let A be a l.s.d. algebra of finite dimension over a field k. Then the Lie algebra $\mathfrak{L}_R(A)$ generated by the right multiplications R_a of A is nilpotent.*

Demonstration. For all elements a and b of A we have $[R_a, R_b] = [L_a, L_b] = L_{[a,b]} = R_{[a,b]} + \mathrm{ad}_{[a,b]}$. Thus the Lie algebras $\mathfrak{L}_R(A)$ is formed by the elements of the form $R_a + \mathrm{ad}_b$, where a is an element of A, and b is an element of $\mathfrak{D}(A)$. We require to show that there exists a positive integer i for which the expression

$$(*) \qquad \left(-\left(\mathrm{ad}_{R_a} + \mathrm{ad}_b\right)\right)^i \left(R_x + \mathrm{ad}_y\right) = \left[\cdots\left[R_x + \mathrm{ad}_y, R_a + \mathrm{ad}_b\right], \cdots, R_a + \mathrm{ad}_b\right]$$

is zero for all elements x and a of A, and y and b of $\mathfrak{D}(A)$. As the right-multiplications R_a are derivations of the Lie algebra A, so $\mathfrak{L}_R(A)$ is a sub-algebra of the Lie algebra of derivations of A. Thus $[R_x, \mathrm{ad}_b] = \mathrm{ad}_{R_x(b)}$, $[[R_x, R_a], \mathrm{ad}_b] = \mathrm{ad}_{[R_x, R_a](b)}$, for all elements a, b and x of A, and $R_x(b)$ and $[R_x, R_a](b)$ belonging to $\mathfrak{D}(A)$ for elements b of $\mathfrak{D}(A)$.

These remarks enable us to say that in the expansion of product $(*)$, we find that, apart from the term

$$R^{(i)}_{x,a} = [\cdots[[R_x, R_a], R_a], \cdots, R_a] = \left(-\mathrm{ad}_{R_a}\right)^i (R_x),$$

there are only interior derivations of the Lie algebra A. As the Lie algebra is solvable, the terms which are interior derivations are zero for sufficiently large i for the following two reasons:

(i) the elements of the central nested sequence of the Lie algebra $\mathfrak{D}(A)$ are characteristic ideals of the Lie algebra A; they are therefore bilateral ideals of A,

(2) the central nested sequence of $\mathfrak{D}(A)$ converges to zero.

Concerning the term $R^{(i)}_{x,a}$ we can show by induction that

$$R^{(i)}_{x,a} = \sum_{r=0}^{i} \binom{i-1}{r}(-1)^r R^r_a L_{[x,a]} R_a^{i-r-1} \text{ for } i \geqslant 1.$$

From Lemma 2.4.1 we see that R^r_a is zero on $\mathfrak{D}(A)$ for r sufficiently large, and that for $i-r-1$ sufficiently large the image of R_a^{i-r-1} is contained in $Z(A)$. Thus by 2.3.8 we have $\mathfrak{D}(A).Z(A)=0$. q.e.d.

The following result is an immediate corollary of the two previous lemmas.

2.4.3. Lemma. *Let A be a l.s.d. algebra of finite dimension n over the field k. For an arbitrary element a of A, the Lie ideals $A_0(a)$ and $A_*(a)$ of A given by Lemma* 2.4.1 *are bilateral ideals.*

The following result is a slight modification of a classical result (see [6, p. 39, Theorem 4]) which we present in a form more suitable to our purposes. To prove this result it suffices to follow the lines of the proof in [6].

2.4.4. Lemma. *Let V be a vector space of finite dimension n over a field k, and let B be a set of endomorphisms of V. If the Lie algebra $\mathfrak{L}(B)$ of endomorphisms of W generated by B is nilpotent, then*

$$V = \bigcap_{h\in B} \ker h^n \oplus \sum_{h\in B} \operatorname{Im} h^n,$$

and the subspaces $\bigcap_{h\in B}\ker h^n$ and $\sum_{h\in B}\operatorname{Im} h^n$ are invariant under B.

This being the case, then, in order to examine the bilateral ideals $A_0 = \bigcap_{a\in A}\ker(R_a)^n$ and $A_* = \sum_{a\in A}\operatorname{Im}(R_a)^n$ of A, we will examine two sequence of bilateral ideals of A.

2.4.5. Sequence of kernels of a left-symmetric algebra. Let A be a left-symmetric algebra, and let $N_1 = N(A)$ be the ideal kernel of A (see §I, Definition 3.3). The preimage N_2 of $N(A/A_1)$ by the canonical homomorphism $j_1\colon A \to A/N_1$ of left-symmetric algebras is a nilpotent bilateral ideal of A. This observation enables us to construct an ascending nested sequence of nilpotent bilateral ideals of A, defined as follows: $N_0 = 0$, $N_1 = N(A)$ and N_{i+1} is the preimage of $N(A/N_i)$ by the canonical mapping $J_i\colon A \to A/N_i$ for $i \geqslant 1$.

The sequence of ideals N_i will be called the sequence of kernels of A. We remark that the quotient algebra N_{i+1}/N_i is an algebra with a trivial product since $N_{i+1}A \subset N_i$. Write $N_\infty = \bigcup_{i\geqslant 0} N_i$. Then $N_\infty = N_i$ if and only if $N_i = N_{i+1}$. To study the ideal $N_\infty = N_\infty(A)$ of a l.s.d. algebra of dimension n we make two remarks.

Remarks (1) There exists an element i of $\{0, 1, \cdots, n\}$ such that $N_i = N_{i+1}$ so that $N_i = N_\infty(A)$; thus $N_\infty(A) = N_n$. That is,

$$N_\infty(A) = \{a \in A; \forall a_1, \cdots, a_n \in A, (R_{a_1} R_{a_2} \cdots R_{a_n})(a) = 0\}.$$

(2) $N_i = N_{i+1}$ if and only if A/N_i is associative, commutative and has an identity. In particular, $A/N_\infty(A)$ is an algebra with an identity.

To prove the two remarks it suffices to note that if A is not commutative, then by Lemma 2.4.1, $N(A) \cap Z(\mathfrak{D}(A)) \neq 0$ (see the demonstration of the Corollary 2.3.4. of Theorem 1). On the other hand, if A is associative and commutative, we have a decomposition of A as that in Theorem 1′, [1].

2.4.6. Lemma. *Let A be a* l.s.d. *algebra of finite dimension n over the field k. Then*

(1) *The ideal $N_\infty(A)$ is the intersectiion of the bilateral ideals $A_0(a)$ of A defined in Lemma* 2.4.1.

(2) *A is nilpotent if and only if $A = N_\infty(A)$.*

Before we begin to prove this lemma it is useful to note that there exist left-symmetric nilpotent algebras for which $N(A) = 0$. This is the case for the algebra with basis $\{e_1, e_2, e_3\}$ over k, for which the nontrivial products are the following:

$$e_1e_2 = e_2, \quad e_1e_3 = -e_3, \quad e_2e_3 = e_1, \quad e_3e_2 = e_1,$$

(see [2]).

Demonstration. The second assertion is an immediate consequence of the first. To prove the first, let N' be the bilateral ideal $\bigcap_{a\in A} \ker(R_a)^n$. It is evident that $N_\infty(A) \subset N'$. Suppose for the moment that $N_\infty(A) \neq N'$. Then there exists an element b of N' such that b is not contained in $N_\infty(A)$. Let b' be the image of b in $A/N_\infty(A)$ by the canonical mapping, and let e be the identity element in $A/N_\infty(A)$. Since $N_\infty(A) \neq N'$, so $A/N_\infty(A) \neq 0$ and hence $e \neq 0$. Consider the element a of A, for which the image of a in $A/N_\infty(A)$ is e. Since b belongs to N', so $R_a^n(b) = 0$ and $R_e^n(b') = b' \neq 0$. There is thus a contradiction, and we must conclude that $N' = N_\infty(A)$.

2.4.7. Demonstration of Theorem 1′. *Existence.* The existence of the decomposition of A is verified by taking $A_0 = \bigcap_a \ker(R_a)^n$, $A_* = \bigcap_a \operatorname{Im}(R_a)^n$. Lemmas 2.4.3 and 2.4.4 assure that $A = A_0 \oplus A_*$ and that $A_* \supset \mathfrak{D}(A)$, $A_0 \subset Z(A)$.

Further, Lemma 2.4.6 tells us that $A_0 = N_\infty(A)$, and so A_0 is nilpotent. Thus $N_\infty(A) = A_0$ and $A_* = A/N_\infty(A)$ has an identity (see Remark 2).

Uniqueness. Let $M_i(A)$, $i \in N$, be the descending nested sequence of vector subspaces of A defined in the following manner: $M_0(A) = A$, $M_1(A)$ is the space generated by the images of the endomorphisms R_a, $a \in A$, and in general $M_i(A)$ is the space generated by the images of the endomorphism $R_{a_1} \circ R_{a_2} \circ \cdots \circ R_{a_i}$, where the elements $a_1, a_2, \cdots, a_i$ are from A. From the identity

$$(ab)c - a(bc) = (ba)c - b(ac)$$

which defines the left-symmetric algebra, it results that the $M_i(A)$ are bilateral ideals of A. Taking $M_\infty(A) = \bigcap_{i \geqslant 0} M_i(A)$, the uniqueness of the decomposition of A described by the theorem is assured by the following lemma.

2.4.8. Lemma. *Let A be a left-symmetric algebra of finite dimension over the field k. If A is a direct sum of two bilateral ideals $A = A_0 \oplus A_*$ such that $N_\infty(A_0) = A_0$, and A_* has a right-identity element, then $N_\infty(A) = A_0$ and $M_\infty(A) = A_*$.*

Demonstration. First of all, the fact that $N_\infty(A) = A_0$ (or the equivalent fact that $M_\infty(A_0) = 0$) implies that A_0 is a nilpotent algebra. Also, if A_* has a right-identity element, then trivially $M_\infty(A_*) = A_*$, and so $N_\infty(A_*) = 0$. Finally, since A is a direct sum of two bilateral ideals, we have

$$N_\infty(A) = N_\infty(A_0) + N_\infty(A_*) = N_\infty(A_0),$$
$$M_\infty(A) = M_\infty(A_0) + M_\infty(A_*) = M_\infty(A_*)$$

III. ON THE EXISTENCE OF LEFT-SYMMETRIC DERIVATION STRUCTURES COMPATIBLE WITH THE LIE STRUCTURE

We know that every solvable Lie algebra of dimension $\leqslant 3$ over k is sub-adjacent to a l.s.d. product (1.6(b), §II). Here we give an example which shows that this property is no longer true for algebras of dimension 4.

In Lemma 1.2 below, we consider the structures of 2-solvable Lie algebras with trivial centers. This lemma enables us to show that such algebras are sub-adjacent to a left-symmetric interior derivation product. This facilitates the construction of a Lie algebra which has only one isomorphism class of l.s.d. structures compatible with the Lie structure. By using this algebra we can construct Lie algebras of arbitrary dimension $\geqslant 4$ which are not sub-adjacent to a l.s.d. product.

The proof of Proposition 1.1 illustrates the usefulness of Theorem 1′ in the research of l.s.d. algebraic structures compatible with a given Lie structure. It is evident that a knowledge of the space of l.s.d. products compatible with a given Lie structure constitutes the first part of the study of obstructions to the existence of these products.

1. The existence of a l.s.d. product compatible with a 2-solvable Lie structure with trivial center

1.1. Theorem. *Let A be a Lie algebra over k. If A is 2-solvable (that is, $\mathfrak{D}(A)$ is Abelian) and has trivial center, then A is sub-adjacent to a left-symmetric interior derivation product.*

The theorem is a consequence of the following lemma and Theorem 1.4 of §II.

1.2. Lemma. *Let A be a 2-solvable non-nilpotent Lie algebra of finite dimension n over k. Then we have the following.*

(1) *For elements a of A such that* ad_a *is not nilpotent,*

$$A = \ker(\mathrm{ad}_a)^n \oplus \mathrm{Im}(\mathrm{ad}_a)^n,$$

where $\mathrm{Im}(\mathrm{ad}_a)^n$ *is an Abelian ideal, and* $\ker(\mathrm{ad}_a)^n$ *is a nonzero subalgebra of A.*

(2) *If A has a trivial center, then*

$$A = C \oplus \mathfrak{D}(A),$$

where C is an Abelian Cartan sub-algebra of A.

Demonstration. Since $\mathfrak{D}(A)$ is an Abelian algebra so, for all elements a and b of A, the linear endomorphism $\mathrm{ad}_a \circ \mathrm{ad}_b$ is a derivation of the algebra A. In particular, for every positive integer i we have

$$(1.1) \qquad (\mathrm{ad}_a)^i[x, y] = \left[(\mathrm{ad}_a)^i x, y\right] + \left[x, (\mathrm{ad}_a)^i y\right].$$

Let a be an element of A such that ad_a is a non-nilpotent endomorphism. Then from Fitting's lemma, $A = \ker(\mathrm{ad}_a)^n \oplus \mathrm{Im}(\mathrm{ad}_a)^n$. Since ad_a is a derivation, the subspace $\ker(\mathrm{ad}_a)^n$ is a sub-algebra of A. Further, $\mathrm{Im}(\mathrm{ad}_a)^n$ is an Abelian sub-algebra of A.

Let x belong to $\ker(\mathrm{ad}_a)^n$, and let t belong to $\mathrm{Im}(\mathrm{ad}_a)^n$. Then $[x, t] = [x, (\mathrm{ad}_a)^n y]$ for some element y of A. Thus by identity (1.1)

$$(\mathrm{ad}_a)^n[x, y] = \left[x, (\mathrm{ad}_a)^n y\right] = [x, t],$$

and so $\mathrm{Im}(\mathrm{ad}_a)^n$ is an ideal of A.

To treat the second assertion of the lemma we will commence by recalling the following salient fact: a Cartan sub-algebra C of a Lie algebra A is a nilpotent sub-algebra of A, which coincides with its normalizer in A (if $[x, C] \subset C$, then $x \in C$ for all elements x of A). We know that $\ker(\mathrm{ad}_a)^n$ is a sub-algebra which coincides with its normalizer in A. If e_0 is an element of A, which is chosen such that $\ker(\mathrm{ad}_{e_0})^n$ has the smallest possible dimension, then $\ker(\mathrm{ad}_{e_0})^n$ is a Cartan sub-algebra of A.

We will show that if A is 2-solvable, and $Z(A) = 0$, then $\mathrm{Im}(\mathrm{ad}_{e_0})^n = \mathfrak{D}(A)$. Consider the commutant of $\mathrm{Im}(\mathrm{ad}_{e_0})^n$ in A: $\underline{z} = \{x \in A; [x, \mathrm{Im}(\mathrm{ad}_{e_0})^n] = 0\}$. Evidently, $\mathfrak{D}(A) \subset \underline{z}$. We will now prove that $\underline{z} \subset \mathrm{Im}(\mathrm{ad}_{e_0})^n$ since then we will have $\mathfrak{D}(A) \subset \underline{z} \subset \mathrm{Im}(\mathrm{ad}_{e_0})^n \subset \mathfrak{D}(A)$. Suppose that $\underline{z} \not\subset \mathrm{Im}(\mathrm{ad}_{e_0})^n$, and let $I = \underline{z} \cap \ker(\mathrm{ad}_{e_0})^n$. We will assume for a moment that I is a nontrivial ideal of the nilpotent algebra, $\ker(\mathrm{ad}_{e_0})^n$ then $J = Z(\ker(\mathrm{ad}_{e_0})^n) \cap I \neq 0$. Consider an element b of J such that $b \neq 0$. We have $[b, \mathrm{Im}(\mathrm{ad}_{e_0})^n] = 0$, since $b \in I \subset \underline{z}$. Thus b belongs to $Z(A)$, and $b \neq 0$, which is absurd. Consequently $\underline{z} \subset \mathrm{Im}(\mathrm{ad}_{e_0})^n$.

To complete the demonstration it remains to prove the assumption which we have just made. Let y belong to I, and let b belong to $\ker(\mathrm{ad}_{e_0})^n$. Then $[y, b] \in \ker(\mathrm{ad}_{e_0})^n$ since $\ker(\mathrm{ad}_{e_0})^n$ is a sub-algebra. Further, $[y, b] \in \mathfrak{D}(A) \subset \underline{z}$, and thus $[b, y] \in I$. In addition, for an element y of $\underline{z}$ with $y \notin \mathrm{Im}(\mathrm{ad}_{e_0})^n$, we set $y = x + x'$ with $x \in \ker(\mathrm{ad}_{e_0})^n$ and $x' \in \mathrm{Im}(\mathrm{ad}_{e_0})^n$. Since $x' \in \underline{z}$, so $x = y - x' \neq 0$ belongs to $\underline{z}$.

Demonstration of the theorem. From the lemma, A is a semi-direct product of two Abelian algebras. If $A = A_0 \times A_1$ is such a decomposition, in order to obtain a product of the required type it suffices to take $L_a = \mathrm{ad}_{f(a)}$, where f is the projection on A_1 parallel to A_0.

Observation. If d is an element of A_0, we can obtain a l.s.d. product on A compatible with the Lie structure, by imposing

$$L_{a_0} = 0, \quad L_{a_1} = \mathrm{ad}_{a_1 + d}.$$

2. A Lie algebra sub-adjacent to a unique left-symmetric derivation structure

Let $\underline{K}$ be a Lie algebra over k with trivial center. Suppose further that $\underline{K}$ has a l.s.d. product compatible with its Lie structure. From Theorem 1′ this product is nilpotent and therefore transitive.

Let $\tilde{K}$ be a simply connected Lie group with Lie algebra $\underline{K}$, and let ρ: $\tilde{\underline{K}} \to \underline{K} \times GL(\underline{K})$ be the etale affine representation defined by the expotential of the mapping $a \to (a, L_a)$. As the l.s.d. product on $\underline{K}$ is transitive, so the action of $\tilde{K}$ on $\underline{K}$ defined by ρ is transitive, and thus the set Ω of points of $\underline{K}$ which have open orbits and discrete isotropies is, in fact, just $\underline{K}$. From §II, 1, the left-symmetric product defined by a point of Ω ($= \underline{K}$) is isomorphic to the initial product, that is to say, isomorphic to the product defined by the origin of $\underline{K}$.

Note that the above remarks hold if k is the field of complex numbers or the field of real numbers.

In the following we will construct a 2-solvable Lie algebra with a trivial center, for which there is, up to an isomorphism, only one l.s.d. structure compatible with the Lie product.

Let A be a Lie algebra over k with basis $\{e_0, e_1, e_2, \cdots, e_n\}$, for which the product is defined as follows:

$$\begin{aligned}&[e_i, e_j] = 0,\ i, j \geqslant 1; \quad [e_0, e_i] = \lambda_i e_i,\ i > 0,\\ &\text{with } \lambda_i \in k^* = k - \{0\}, \text{ the } \lambda_i \text{ being pairwise distinct.}\end{aligned} \tag{1}$$

From Theorem 1.1, A is sub-adjacent to a left-symmetric interior product. We will study the space of the l.s.d. products compatible with the Lie structure.

For $0 \leqslant j \leqslant n$, we set $D(e_j) = \sum_{i=0}^{n} \alpha_{ij} e_i$, with $\alpha_{ij} \in k$. If D is a derivation of the algebra A, then evidently $\alpha_{0j} = 0$ when $j > 0$. As well, $[D(e_0), e_j] + [e_0, D(e_j)] = \lambda_j D(e_j)$ implies that

$$\alpha_{00}\lambda_j e_j + \sum_{i=1}^{n} \alpha_{ij}\lambda_i e_i = \sum_{i=0}^{n} \lambda_j \alpha_{ij} e_i,$$

or, equivalently,

$$(\alpha_{00}\lambda_j + \alpha_{jj}\lambda_j)e_j + \sum_{i\neq j} \alpha_{ij}\lambda_i e_i = \lambda_j \alpha_{jj} e_j + \sum_{i \neq j} \alpha_{ij}\lambda_j e_i,$$

and so $\alpha_{00} = 0$, and $\alpha_{ij} = 0$ for $i \neq j$. Now suppose that the relations $L_{e_i} e_j = \sum_{l=0} \alpha^i_{lj} e_l$ define a l.s.d. product compatible with the Lie structure. Since $L_{e_0} e_i - L_{e_i} e_0 = \lambda_i e_i$ so $\alpha^0_{ii} e_i - \sum_{l=1}^{n} \alpha^i_{l0} e_i = \lambda_i e_i$, that is, for $i > 0$ we have $\alpha^0_{ii} - \alpha^i_{i0} = \lambda_i$; $\alpha^i_{l0} = 0$ for $l \geqslant 1$, $l \neq i$. Further, for $j \geqslant 1$ we have $\alpha^i_{jj} e_j = \alpha^j_{ii} e_i$, and so $\alpha^i_{jj} = 0$ for $i \neq j$. Thus there remains

$$L_{e_i} e_0 = \alpha^i_{i0} e_i;\ L_{e_i} e_i = \alpha^i_{ii} e_i;\ L_{e_i} e_j = 0, j \neq i$$

with

$$\alpha^i_{ii} - \alpha^i_{i0} = \lambda_i. \tag{2}$$

On the other hand, the relations $[L_{e_0}, L_{e_i}] = \lambda_i L_{e_i}$ tell us that for $i > 0$

$$\alpha^0_{ii}\alpha^i_{ii} - \alpha^i_{ii}\alpha^0_{ii} = \lambda_i \alpha^i_{ii},$$

$$\alpha^0_{ii}\alpha^i_{i0} - \alpha^i_{ii}\alpha^0_{i0} = \lambda_i \alpha^i_{i0}.$$

Then, since $\lambda_i \neq 0$, $\alpha^i_{ii} = 0$, and thus the second relation and (2) imply that $\alpha^i_{i0} = 0$.

In summary, every l.s.d. product compatible with the Lie structure of A can be written as follows:

$$(3)\qquad L_{e_0}(e_0) = \sum_{l=1}^{n} \alpha_l e_l; \qquad L_{e_0}(e_i) = \lambda_i e_i; \qquad L_{e_i} = 0, i > 0,$$

where the α_i are elements of k.

We remark that $Le_0 = \mathrm{ad}_b$, where $b = e_0 + \Sigma_{l=1}^{n}(-\alpha_l/\lambda_l)e_l$, so every product is defined by interior derivations of the Lie algebra A. Note also that the space of these products can be identified with the space of endomorphisms f of the Lie algebra A defined by

$$f(e_0) = e_0 + \sum_{i=1}^{n} \beta_i e_i; f(e_j) = 0 \quad \text{for } j > 0.$$

The following propositiion shows that these products are isomorphic.

2.1. Proposition. *The Lie algebra defined by* (1) *has, up to an isomorphism, only one structure of a left-symmetric derivation algebra compatible with the Lie structure, which is given by* (3).

Demonstration. It suffices to show that the following two left-symmetric structures are isomorphic:

$$e_0 e_0 = 0, e_0 e_i = \lambda_i e_i, e_i e_j = 0;$$

$$e_0 * e_0 = \sum_{i=1}^{n} \alpha_i e_i; e_0 * e_i = \lambda_i e_i, e_i * e_j = 0.$$

Consider the linear isomorphism p of A defined by

$$p(e_0) = e_0 + \sum_{i=1}^{n} \left(-\frac{\alpha_i}{\lambda_i}\right) e_i; \qquad p(e_j) = e_j, \qquad j > 0.$$

Since we have

$$\begin{aligned} p(e_0) * p(e_0) &= e_0 * e_0 + \sum_{i=1}^{n} \left(-\frac{\alpha_i}{\lambda_i}\right) e_0 * e_i \\ &= \sum_{i=1}^{n} \alpha_i e_i + \sum_{i=1}^{n} \left(-\frac{\alpha_i}{\lambda_i}\right) \lambda_i e_i \\ &= p(e_0 e_0), \end{aligned}$$

p is an isomorphism of left-symmetric algebras. q.e.d.

Note that the couple (p, v), where $v = \Sigma_{i=1}^{n}(-\alpha_i/\lambda_i)e_i$, defines an isomorphism of the affine space A, for which

$$L'_a = p \circ L_a \circ p^{-1}, \quad a = p(a) - L_a(v)$$

for all elements a of A, L_a and L'_a being the left-multiplications associated respectively to the products considered above. Consequently (p, v) is an isomorphism between the affine representations

$$a \to (a, L_a), \quad a \to (a, L'_a).$$

2.2. Corollary. *If A' is a Lie algebra of finite dimension $n + 2 > 3$, over k, which is a nontrivial central extension of the algebra A defined by* (1), *then A' is not sub-adjacent to l.s.d. product.*

Demonstration. The fact that the extension is nontrivial implies that there are integers i and j such that $[e_i, e_j] \neq 0$.

Consider now the exact sequence of Lie algebras: $0 \to ke_{n+1} \to A' \to A \to 0$, where ke_{n+1} is the central ideal of A'.

Suppose that the Lie algebra A' is sub-adjacent to a l.s.d. product. From Theorem 1′, this product satisfies the condition $L'_{e_{n+1}} = R'_{e_{n+1}} = 0$. Thus with the obvious notation, the l.s.d. product of A' is written:

$$\begin{aligned}(m, x)(n, y) &= (m,0)(n, y) + (0, x)(n,0) + (0, x)(0, y)\\ &= (0, x)(0, y) = \big(\alpha(x, y)e_{n+1}, xy\big),\end{aligned}$$

where $\alpha(x, y)$ belongs to k, and xy denotes a l.s.d. product on A. Moreover, we have $\operatorname{Im} L'_{(m,x)} \subset Z(A')$ for all elements (m, x) of A' such that $L_x = 0$. In particular, $\operatorname{Im} L'_{e_l} \subset ke_{n+1}$ for $l = 1, 2, \cdots, n$, and so, as L'_{e_l} is a derivation, we have

$$L'_{e_i}(d) = L'_{e_j}(d) = 0 \quad \text{for all elements } d \text{ of } \mathfrak{D}(A).$$

Consequently, $\mathfrak{D}(A') \supset \{e_i, e_j\}$ and $L'_{e_i}e_j - L'_{e_j}e_i \neq 0$, which is a contradiction.

2.3. Examples. (1) Let A' be the Lie algebra over k with basis $\{e_0, e_1, e_2, e_3\}$ defined as follows:

$$[e_0, e_1] = \lambda e_1, \quad [e_0, e_2] = -\lambda e_2, \quad [e_1, e_2] = \beta e_3, \quad [e_3, A'] = 0,$$

where λ and β are nonzero elements of k.

The algebra A' is a nontrivial central extension of the algebra A given by

$$[e_1, e_2] = 0, \quad [e_0, e_1] = \lambda e_1, \quad [e_0, e_2] = -\lambda e_2.$$

The algebra A is 2-solvable and has trivial center. From Corollary 2.2, A' is not sub-adjacent to a l.s.d. product.

(2) Let A a noncommutative algebra, suppose that the bracket $[a, b]$ is always equal to a linear combination of a and b. It is shown that A has this property if and only if there exist a commutative ideal I of codimension 1 and an element $e_0 \notin I$ such that $[e_0, a] = a$ for every $a \in I$. Proposition 2.1 shows that the Lie algebra has only one structure of left-symmetric derivation algebra

compatible with the Lie structure, which is given by

$$L_{e_0} = \mathrm{ad}_{e_0}, L_{e_i} = 0, i \geqslant 1,$$

where $\{e_0, e_1, \cdots, e_n\}$ is a basis of A.

Added in Proof. After this paper was written, the author knew a counter-example given by D. Fried to the Auslander's conjecture.

Bibliography

[1] A. A. Albert, *Structure of algebras*, Colloq. Publ., Amer. Math. Soc., 1939.

[2] L. Auslander, *Simply transitive groups of affine motions*, Amer. J. Math. **99** (1977) 809–826.

[3] N. Bourbaki, *Groupes et algèbres de Lie*, Hermann, Paris, 1960, 1972, Chaps. I, III.

[4] G. Giraud & A. Medina, *Existence de certaines connexions plates invariants sur les groupes de Lie*, Ann. Inst. Fourier (Grenoble) **27** (1977) 233–245.

[5] J. Helmstetter, *Radical et groupe formel d'une algèbre symétrique à gauche*, Thèse Troisième Cycle, Grenoble, 1975.

[6] N. Jacobson, *Lie algebras*, Interscience, New York, 1962.

[7] J. L. Koszul, *Domaines bornés homogènes et orbites de groupes de transformations affines*, Bull. Soc. Math. France **89** (1961) 515–533.

[8] S. Kobayashi & K. Nomizu, *Foundations of differential geometry*, Vol. I, Interscience, New York, 1963.

[9] A. Medina, *Sur quelques algèbres symétriques à gauche dont l'algèbre de Lie sous-jacente est résolvable*, C. R. Acad. Sci. Paris **286** (1978) 173–176.

[10] ______, *Sur quelques groupes de Lie de transformations affines*, C. R. Acad. Sci. Paris **287** (1978) 339–342.

[11] ______, *Autour des connexions plates invariantes à gauche sur les groupes de Lie*, Thèse d'Etat, Montpellier, 1979.

[12] J. Milnor, *On fundamental groups of complete affinely flat manifolds*, Advance in Math. **25** (1977) 178–187.

[13] P. Molino, *Sur quelques propriétés des G-structures*, J. Differential Geometry **7** (1972) 489–518.

[14] J. Scheuneman, *Translations in certains groups of affine motions*, Proc. Amer. Math. Soc. **47** (1977) 223–228.

[15] I. M. Singer & S. Sternberg, *The infinite groups of Lie and Cartan*, J. Analyse Math. **15** (1965) 1–114.

[16] E. B. Vinberg, *Convex homogeneous cones*, Transl. of Moscow Math. Soc. No. 12, 340–403.

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